ABBOTSLEIGH

AUGUST 2010
YEAR 12
ASSESSMENT 4
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.

Outcomes assessed

HSC course

- **E1** appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
- **E2** chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
- **E3** uses the relationship between algebraic and geometric representations of complex numbers and of conic sections
- **E4** uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
- **E5** uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces and resisted motion
- **E6** combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
- **E7** uses the techniques of slicing and cylindrical shells to determine volumes
- **E8** applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems
- **E9** communicates abstract ideas and relationships using appropriate notation and logical argument

Harder applications of the Extension 1 Mathematics course are included in this course. Thus the Outcomes from the Extension 1 Mathematics course are included.

From the Extension 1 Mathematics Course

Preliminary course

- **PE1** appreciates the role of mathematics in the solution of practical problems
- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives that require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

- **HE1** appreciates interrelationships between ideas drawn from different areas of mathematics
- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion or exponential growth and decay
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

QUESTION 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \frac{x^2}{\left(1+x^3\right)^2}\,dx$. **2**

(b) Find $\int \frac{x^2+4}{x^2+1}\,dx$. **2**

(c) Use integration by parts to evaluate $\int_0^1 x e^{-3x}\,dx$. **3**

(d) (i) Find real numbers a, b and c such that

$$\frac{x}{(x-1)^2(x-2)} \equiv \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x-2}.$$ **2**

(ii) Evaluate $\int \frac{x}{(x-1)^2(x-2)}\,dx$. **2**

(e) Use the substitution $x = \sin\theta$ to evaluate $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$. **4**

Marks

QUESTION 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 3 - i$ and $w = 2 + i$. Express the following in the form $x + iy$, where x and y are real numbers:

(i) $\dfrac{z}{w}$ **2**

(ii) $\overline{-2iz}$ **2**

(b) Let $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.

(i) Express z in modulus-argument form. **2**

(ii) Show that $z^6 = 1$. **2**

(iii) Hence, or otherwise, graph all the roots of $z^6 - 1 = 0$ on an Argand diagram. **2**

(c) The complex numbers α, β, γ and δ are represented on an Argand diagram by the points A, B, C and D respectively.

(i) Describe the point that represents $\dfrac{1}{2}(\alpha + \gamma)$. **1**

(ii) Deduce that if $\alpha + \gamma = \beta + \delta$ then $ABCD$ is a parallelogram. **2**

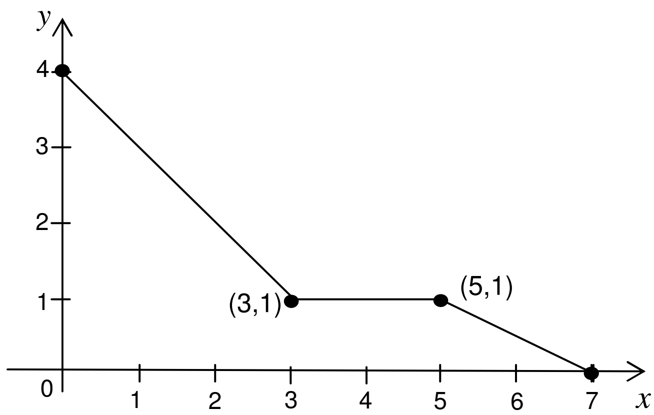
(d) Let $z = x + iy$. Find the points of intersection of the curves given by:

$|z - i| = 1$ and $\mathrm{Re}(z) = \mathrm{Im}(z)$. **2**

Marks

QUESTION 3 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows the graph of the function $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = f(|x|)$ **2**

(ii) $y = f(2-x)$ **2**

(iii) $y = \log_e f(x)$. **2**

(b) Sketch the graph of $y = \dfrac{1}{x(x-2)}$, without the use of calculus. **3**

(c) (i) Find the value of g for which $P(x) = 9x^4 - 25x^2 + 10gx - g^2$ is divisible by both $x-1$ and $x+2$. **3**

(ii) With this value of g, solve the equation $9x^4 - 25x^2 + 10gx - g^2 = 0$. **3**

Marks

QUESTION 4 (15 marks) Use a SEPARATE writing booklet.

(a) The area bounded by the curve $y = x^2 + 2$ and the line $y = 4 - x$ is rotated about the line $y = 1$.

(i) Find the points of intersection of the two curves. **2**

(ii) By considering slices perpendicular to the x axis, show that the area, $A(x)$ of a typical slice is given by:

$$A(x) = \pi\left(8 - 6x - x^2 - x^4\right).$$ **2**

(iii) Find the volume of the solid formed. **2**

(b) Show that for all real x, $0 < \dfrac{1}{x^2 + 2x + 2} \leq 1$. **3**

(c) (i) If $I_n = \int x^3 \left(\log_e x\right)^n dx$, show that $I_n = \dfrac{x^4}{4}\left(\log_e x\right)^n - \dfrac{n}{4} I_{n-1}$. **3**

(ii) Hence, or otherwise, evaluate $\int_1^2 x^3 \left(\log_e x\right)^2 dx$. **3**

Marks

QUESTION 5 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Factorise the polynomial $z^3 - 1$ over the rational field. **1**

(ii) If w is a complex root of 1, show that $1 + w + w^2 = 0$. **1**

(iii) Hence, or otherwise, simplify $(1+w^2)(1+w^4)(1+w^8)(1+w^{10})$. **2**

(b) Prove that if $a \neq c$ there are always two real values of k which will make $ax^2 + 2bx + c + k(x^2+1)$ a perfect square. **3**

(c) The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are two variable points on the hyperbola $xy = c^2$ which move so that the points P, Q and $S\left(c\sqrt{2}, c\sqrt{2}\right)$ are always collinear. The tangents to the hyperbola at P and Q meet at the point R.

(i) Show that the equation of the chord PQ is $x + pqy = c(p+q)$ **2**

(ii) Hence show that $p + q = \sqrt{2}(1 + pq)$. **1**

(iii) Show that R is the point $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. You may assume that the tangent at any point $T\left(ct, \frac{c}{t}\right)$ has equation $x + t^2 y = 2ct$. (Do NOT prove this) **3**

(iv) Hence find the equation of the locus of R. **2**

Marks

QUESTION 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove that if x and y are positive numbers then $(x+y)^2 \geq 4xy$. **2**

(ii) Deduce that if a, b, c and d are positive numbers then

$$\frac{1}{4}(a+b+c+d)^2 \geq ac+ad+bc+bd .$$ **2**

(b) Scientists use a pressure gauge which measures depth as it sinks towards the ocean floor. The gauge of mass 2 kg is released from rest at the ocean's surface. As it sinks in a vertical line, the water exerts a resistance to its motion of $4v$ Newtons, where v ms^{-1} is the velocity of the gauge.

Let x be the displacement of the ball measured vertically downwards from the ocean's surface, t be the time in seconds elapsed after the gauge is released, and g be the constant acceleration due to gravity.

(i) Show that $\frac{d^2x}{dt^2} = g - 2v$. **2**

(ii) Hence show that $t = \frac{1}{2}\log_e\left(\frac{g}{g-2v}\right)$. **3**

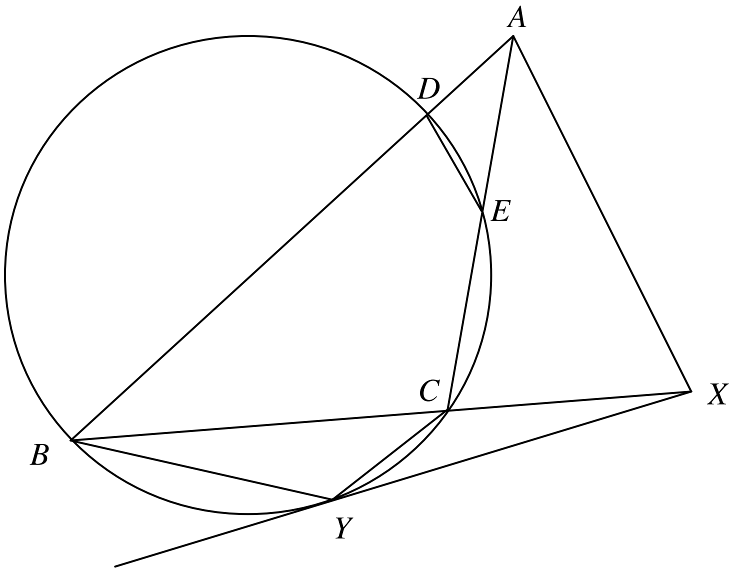
(iii) Show that $v = \frac{g}{2}\left(1 - e^{-2t}\right)$. **2**

(iv) Write down the limiting (terminal) velocity of the gauge. **1**

(v) At a particular location, the gauge takes 180 seconds to hit the ocean floor. Using $g = 10\,\text{ms}^{-2}$, calculate the depth of the ocean at that location, giving your answer correct to the nearest metre. **3**

QUESTION 7 (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram XY is a tangent to the circle and $XY = XA$.



(i) Show that $\Delta XCY \parallel\!| \Delta XBY$. **2**

(ii) Hence explain why $\dfrac{XY}{BX} = \dfrac{CX}{XY}$. **1**

(iii) Show that $\Delta AXC \parallel\!| \Delta AXB$. **3**

(iv) Prove that $DE \parallel AX$. **2**

(b) Consider the function $y = f(x)$ in the interval $1 \le x \le n$.

(i) Sketch a possible graph of $y = f(x)$ given $f(x) \ge 0$ and $f''(x) < 0$. **1**

(ii) Show, by comparing the area under the curve $y = f(x)$ between $x = 1$ and $x = n$, with the area of a region found using repeated applications of the Trapezoidal Rule, each of width 1 unit, that

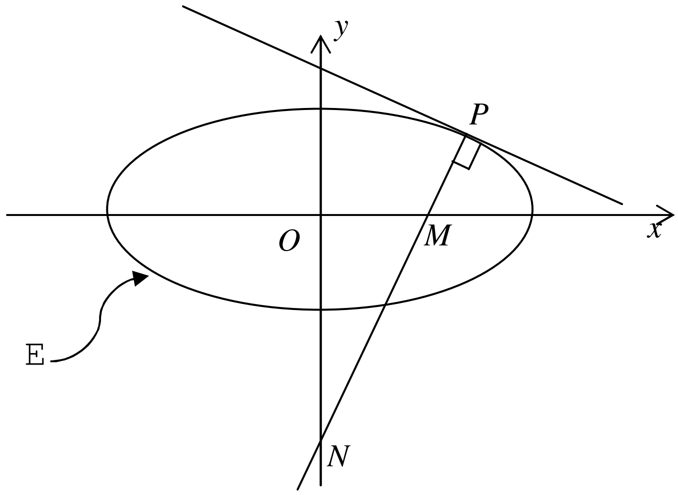
$$\int_1^n f(x)\,dx > \frac{1}{2}f(1) + \frac{1}{2}f(n) + \sum_{r=2}^{n-1} f(r).$$ **2**

(iii) By taking $f(x) = \log_e x$ in the inequality from (b) part (ii) above, deduce that if n is a positive integer, then

$$n! < n^{n+\frac{1}{2}} e^{-n+1}.$$ **4**

QUESTION 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The ellipse E has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse.

(i) Show that the equation of the normal to the ellipse at P is $y-b\sin\theta=\frac{a\sin\theta}{b\cos\theta}(x-a\cos\theta)$. **2**

(ii) The normal at P meets the x axis at M and the y axis at N as shown in the diagram above.
Prove that $\frac{PM}{PN}=1-e^2$ where e is the eccentricity of E. **3**

(b) If $A(x)=\frac{1}{2}+\frac{1}{3}\binom{n}{1}x+\frac{1}{4}\binom{n}{2}x^2+\text{.................}+\frac{1}{n+2}x^n$,

(i) Show that $\frac{d}{dx}\left\{x^2A(x)\right\}=x(1+x)^n$. **3**

(ii) Show that $x(1+x)^n=(1+x)^{n+1}-(1+x)^n$. **1**

(iii) Hence show that $x^2A(x)=\frac{(1+x)^{n+2}-1}{n+2}-\frac{(1+x)^{n+1}-1}{n+1}$. **3**

(iv) Deduce that $\sum_{r=0}^{n}\frac{1}{r+2}\binom{n}{r}=\frac{n\cdot 2^{n+1}+1}{(n+1)(n+2)}$. **3**

End of paper

ABBOTSLEIGH EXTENSION 2 TRIAL 2010 SOLUTIONS

QUESTION 1

(a) $\int \frac{x^2}{(1+x^3)^2}\,dx = \frac{1}{3}\int 3x^2(1+x^3)^{-2}\,dx$

$= \frac{1}{3}\frac{(1+x^3)^{-1}}{-1} + C$

$= \frac{-1}{3(1+x^3)} + C$

(b) $\int \frac{x^2+4}{x^2+1}\,dx = \int \frac{x^2+1+3}{x^2+1}\,dx$

$= \int \left(1 + \frac{3}{x^2+1}\right) dx$

$= x + 3\tan^{-1}x + C$

(c) Let $I = \int_0^1 x e^{-3x}\,dx$

Let $u = x$ $\quad dv = e^{-3x}\,dx$

$du = dx$ $\quad v = -\frac{1}{3}e^{-3x}$

$\therefore I = \left[x \times -\frac{1}{3}e^{-3x}\right]_0^1 - \int_0^1 -\frac{1}{3}e^{-3x}\,dx$

$= -\frac{1}{3}e^{-3} + 0 + \left[\frac{1}{3} \times \frac{e^{-3x}}{-3}\right]_0^1$

$= \frac{-1}{3e^3} + \frac{1}{-9}\left(e^{-3} - e^0\right)$

$= \frac{1}{9} \times \left(\frac{-1}{e^3} - \frac{3}{e^3} + 1\right)$

$= \frac{e^3 - 4}{9e^3}$

(d) (i) $x = a(x-1)(x-2) + b(x-2) + c(x-1)^2$

Let $x = 1$, $\quad 1 = -b \quad \Rightarrow \quad b = -1$

Let $x = 2$, $\quad 2 = c \quad \Rightarrow \quad c = 2$

Let $x = 0$, $\quad 0 = a \times 2 - 1 \times -2 + 2 \times 1 \Rightarrow$

$\therefore a = -2$

(ii) $\therefore \int \frac{x\,dx}{(x-1)^2(x-2)} = \int \left\{\frac{-2}{x-1} - \frac{1}{(x-1)^2} + \frac{2}{x-2}\right\} dx$

$= -2\ln(x-1) + \frac{1}{x-1} + 2\ln(x-2)$

$= \frac{1}{x-1} + 2\ln\left(\frac{x-2}{x-1}\right) + C$

Q1(e) Let $I = \int_{-1/2}^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$

$= 2\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$ (even fn)

Let $x = \sin\theta$ $\qquad x = \frac{1}{2}, \quad \theta = \frac{\pi}{6}$

$dx = \cos\theta\,d\theta$ $\qquad x = 0, \quad \theta = 0$

$\therefore I = 2\int_0^{\pi/6} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \times \cos\theta\,d\theta$

$= 2\int_0^{\pi/6} \frac{\sin^2\theta}{\cos\theta} \times \cos\theta\,d\theta$

$= 2\int_0^{\pi/6} \frac{1}{2}(1 - \cos 2\theta)\,d\theta$

$= \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/6}$

$= \left(\frac{\pi}{6} - \frac{\sin \pi/3}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)$

$= \frac{\pi}{6} - \frac{\sqrt{3}/2}{2} - 0$

$= \frac{1}{12}\left(2\pi - 3\sqrt{3}\right)$

Q2 (a)

(i) $\frac{z}{w} = \frac{3-i}{2+i} \times \frac{2-i}{2-i}$

$= \frac{6-5i-1}{4+1}$

$= \frac{5-5i}{5}$

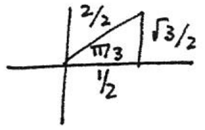
$= 1-i$

(ii) $\overline{-2iz} = \overline{-2i(3-i)}$

$= \overline{(-6i-2)}$

$= -2+6i$

(b)



(i) $z = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

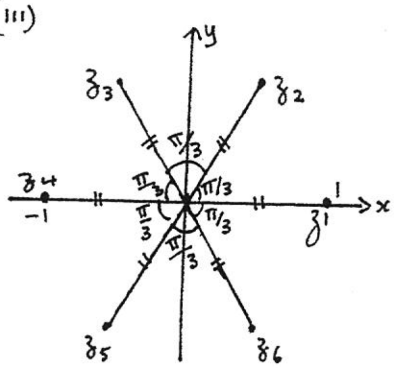
(ii) $z^6 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^6$

$= \cos 2\pi + i\sin 2\pi$

$= 1 + i \times 0$

$= 1$

(iii)



(c) (i) $\frac{1}{2}(\alpha+\gamma)$ represents the midpoint of AC.

(ii) if $\alpha+\gamma = \beta+\delta$

then $\frac{\alpha+\gamma}{2} = \frac{\beta+\delta}{2}$

∴ midpoint of AC = midpoint of BD

∴ ABCD is a parallelogram since its diagonals bisect each other.

Q2 (d) $|z-i| = 1$ represents the circle $x^2+(y-1)^2 = 1$ ①

$Re(z) = Im(z)$ represents the line $x = y$ ②

Substitute ② into ①

$y^2 + (y-1)^2 = 1$

$y^2 + y^2 - 2y + 1 = 1$

$2y^2 - 2y = 0$

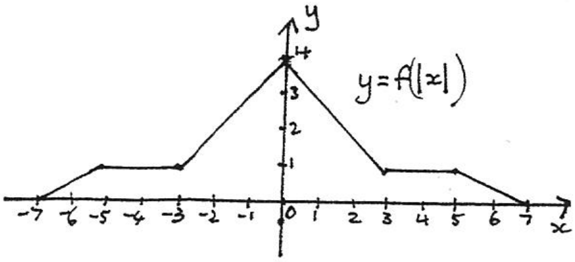
$y(y-1) = 0$

$y = 0$ or 1

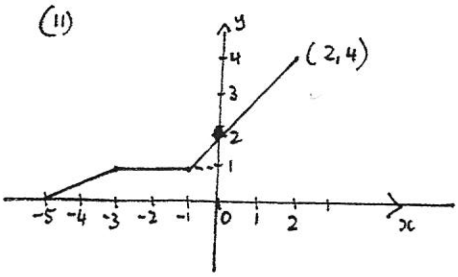
$y=0,\ x=0$ or $y=1,\ x=1$

∴ Pts of intersection are $(0,0)$ and $(1,1)$

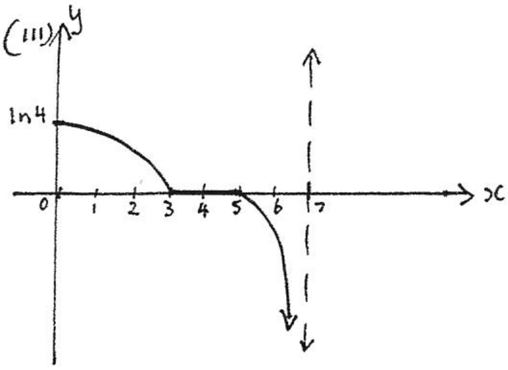
Q 3 (a) (i)



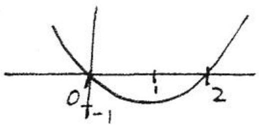
(ii)



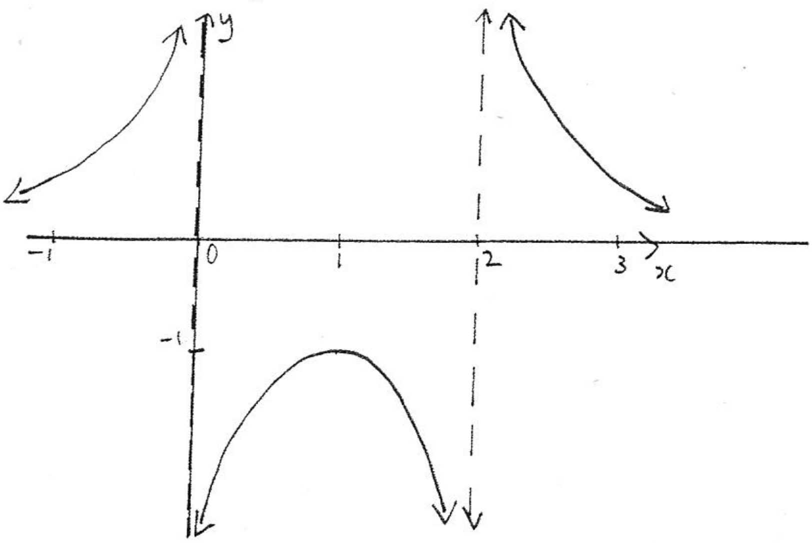
(iii)



(b) $y = x(x-2)$



$$y = \frac{1}{x(x-2)}$$



Q3 (c) (i) $P(1) = P(-2) = 0$

$\therefore 9 - 25 + 10g - g^2 = 0$

$g^2 - 10g + 16 = 0$

$(g-8)(g-2) = 0$

$\therefore g = 8$ or 2

also $144 - 100 - 20g - g^2 = 0$

$g^2 + 20g - 44 = 0$

$(g+22)(g-2) = 0$

$g = -22$ or 2

$\therefore P(1) = P(-2) = 0$ only if $g = 2$

(ii) $9x^4 - 25x^2 + 20x - 4 = (x-1)(x+2)R(x)$

$\therefore \quad = (x^2 + x - 2)(9x^2 + mx + 2)$

x^3 term: $0 = 9 + m$

$\therefore m = -9$

$\therefore P(x) = (x-1)(x+2)(9x^2 - 9x + 2)$
$= (x-1)(x+2)(3x-1)(3x-2)$

$\therefore$ Solutions are $x = 1, -2, \frac{1}{3}, \frac{2}{3}$

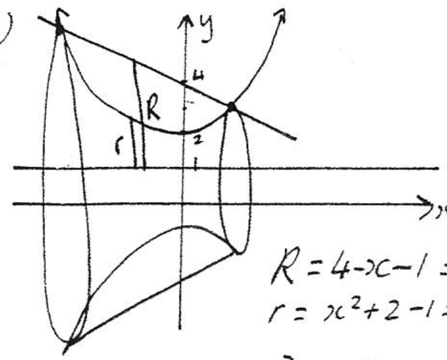
Q4(a)(i) $x^2+2=4-x$

$x^2+x-2=0$

$(x+2)(x-1)=0$

∴ Pts of intersection are $(1,3)$ and $(-2,6)$

(ii)



$R = 4-x-1 = 3-x$

$r = x^2+2-1 = x^2+1$

Area of slice $= \pi R^2 - \pi r^2$

$A(x) = \pi(3-x)^2 - \pi(x^2+1)^2$

$= \pi\{9-6x+x^2-x^4-2x^2-1\}$

$= \pi\{8-6x-x^2-x^4\}$

(iii) Vol of slice $= A(x)\,\delta x$

Vol of solid $= \sum_{x=-2}^{1} \pi(8-6x-x^2-x^4)\,\delta x$

$= \pi\int_{-2}^{1}(8-6x-x^2-x^4)\,dx$

$= \pi\left[8x-3x^2-\frac{x^3}{3}-\frac{x^5}{5}\right]_{-2}^{1}$

$= \pi\left(8-3-\frac{1}{3}-\frac{1}{5}\right) - \pi\left(-16-12+\frac{8}{3}+\frac{32}{5}\right)$

$= \pi\left(5-\frac{8}{15}+28-\frac{136}{15}\right)$

$= \frac{117\pi}{5}$ cubic units

b) $x^2+2x+2 = (x+1)^2+1$

min value of x^2+2x+2 is 1

∴ max value of $\frac{1}{x^2+2x+2}$ is 1

Also, $\lim_{x\to\infty} \frac{1/x^2}{1+2/x+2/x^2} = \frac{0}{1+0+0}$

$= 0$

∴ min value of $\frac{1}{x^2+2x+2} = 0$ (as a limit)

∴ $0 < \frac{1}{x^2+2x+2} \le 1$

4(c)

(i) $I_n = \int x^3(\log_e x)^n\,dx$ let $u=(\log_e x)^n$, $du = n(\log_e x)^{n-1}\cdot\frac{1}{x}\cdot dx$, $dv = x^3\,dx$, $V = \frac{x^4}{4}$

$\therefore I_n = (\log_e x)^n \times \frac{x^4}{4} - \int \frac{x^4}{4}\times n(\log_e x)^{n-1}\times\frac{1}{x}\,dx$

$= \frac{x^4}{4}(\log_e x)^n - \frac{n}{4}\int x^3(\log_e x)^{n-1}\,dx$

$I_n = \frac{x^4}{4}(\log_e x)^n - \frac{n}{4}I_{n-1}$

(ii) $I_2 = \left[\frac{x^4}{4}(\log_e x)^2\right]_1^2 - \frac{1}{2}I_1$

$= \frac{16}{4}(\log_e 2)^2 - 0 - \frac{1}{2}\left\{\left[\frac{x^4}{4}(\log_e x)\right]_1^2 - \frac{1}{4}I_0\right\}$

$= 4(\log_e 2)^2 - \frac{1}{8}\left(16(\log_e 2) - 0\right) + \frac{1}{8}\int_1^2 x^3\,dx$

$= 4(\log_e 2)^2 - 2\log_e 2 + \left[\frac{x^4}{32}\right]_1^2$

$= 4(\log_e 2)^2 - 2\log_e 2 + \frac{1}{2} - \frac{1}{32}$

$= 4(\log_e 2)^2 - 2\log_e 2 + \frac{15}{32}$

Q5 (a) (i) $z^3-1=(z-1)(z^2+z+1)$

(ii) If ω is a complex root then

$\omega-1=0 \Rightarrow \omega=1$ is not complex

$\therefore \omega^2+\omega+1=0$

also $\omega^3=1$

(iii) $(1+\omega^2)(1+\omega)(1+\omega^8)(1+\omega^{10})$

$=(-\omega)(1+\omega\cdot\omega^3)(1+(\omega^3)^2\cdot\omega^2)(1+(\omega^3)^3\cdot\omega)$

$=(-\omega)(1+\omega)(1+\omega^2)(1+\omega)$

$=-\omega\times-\omega^2\times-\omega\times-\omega^2$

$=\omega^3\times\omega^3$

$=1$

(b) Re-arranging quadratic,

$ax^2+2bx+c+k(x^2+1)$

$=x^2(a+k)+2bx+(c+k)$

For a p.s., $\Delta=0$

$4b^2-4(a+k)(c+k)=0$

$b^2-ac-ak-ck-k^2=0$

$k^2+k(a+c)+ac-b^2=0$

If there are 2 values of k, this $\Delta>0$

$\Delta=(a+c)^2-4(ac-b^2)$

$=a^2+2ac+c^2-4ac+4b^2$

$=a^2-2ac+c^2+4b^2$

$=(a-c)^2+4b^2$

which is always >0 if $a\neq c$

$\therefore$ There are always 2 values of k which make the quadratic a perfect square.

5 (c) (i) $P(cp,\frac{c}{p})$ $Q(cq,\frac{c}{q})$

$\frac{y-\frac{c}{p}}{x-cp}=\frac{\frac{c}{q}-\frac{c}{p}}{cq-cp}$

$\therefore \quad =\frac{\frac{1}{q}-\frac{1}{p}}{q-p}\times\frac{pq}{pq}$

$=\frac{p-q}{pq(q-p)}$

$=-\frac{1}{pq}$

$\therefore pqy-cq=-x+cp$

$\therefore x+pqy=c(p+q)$

is the equation of PQ

(ii) since $S(c\sqrt{2},c\sqrt{2})$ lies on PQ,

$c\sqrt{2}+pq\times c\sqrt{2}=c(p+q)$

$\therefore \sqrt{2}(1+pq)=p+q$

(iii) tangent at P is $x+p^2y=2cp$ ①

tangent at Q is $x+q^2y=2cq$ ②

① − ② $\quad y(p^2-q^2)=2c(p-q)$

$y=\frac{2c(p-q)}{(p-q)(p+q)}$

$\therefore y=\frac{2c}{p+q}$ into ①

$x+p^2\times\frac{2c}{p+q}=2cp$

$x=2cp-\frac{2cp^2}{p+q}$

$=\frac{2cp^2+2cpq-2cp^2}{p+q}$

$=\frac{2cpq}{p+q}$

$\therefore R\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right)$

(iv) $x=\frac{2cpq}{p+q}$ ① $\quad y=\frac{2c}{p+q}$ ②

① + ② $\quad x+y=\frac{2c}{p+q}(pq+1)$

$x+y=\frac{2c}{p+q}\times\frac{p+q}{\sqrt{2}}$ (from (ii))

$=\frac{2c}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$

$\therefore x+y=c\sqrt{2}$

Q6 (a) (i) $(x-y)^2 \geq 0$

$x^2 - 2xy + y^2 \geq 0$

$x^2 + y^2 \geq 2xy$

$x^2 + 2xy + y^2 \geq 4xy$

$\therefore \quad (x+y)^2 \geq 4xy$

(ii) let $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$

$\therefore \left(\frac{a+b+c+d}{2}\right)^2 \geq 4\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)$

$\frac{1}{4}(a+b+c+d)^2 \geq ac+ad+bc+bd$

(b) (i) $F = ma$

$2g - 4v = 2\times\frac{d^2x}{dt^2}$

$\therefore \quad \frac{d^2x}{dt^2} = g - 2v$

(ii) $\therefore \frac{dv}{dt} = g - 2v$

$\frac{dt}{dv} = \frac{1}{g-2v}$

$t = \int_0^v \frac{1}{g-2v}\,dv$ since starts at rest.

$= \left[-\frac{1}{2}\ln(g-2v)\right]_0^v$

$= -\frac{1}{2}\ln(g-2v) + \frac{1}{2}\ln g$

$\therefore t = \frac{1}{2}\ln\frac{g}{g-2v}$

(iii) From (ii) $2t = \ln\frac{g}{g-2v}$

$e^{2t} = \frac{g}{g-2v}$

$\frac{g-2v}{g} = e^{-2t}$

$g - 2v = ge^{-2t}$

$2v = g(1-e^{-2t})$

$\therefore v = \frac{g}{2}(1-e^{-2t})$

(iv) From (iii) as $t \to \infty$ $e^{-2t} \to 0$

$\therefore v \to \frac{g}{2}$ m/s (terminal velocity)

(v) From (iii) $\frac{dx}{dt} = \frac{g}{2}(1-e^{-2t})$, $g = 10$

$x = 5\int_0^{180}(1-e^{-2t})\,dt$

$= \left[5t + \frac{5e^{-2t}}{2}\right]_0^{180}$

$= 5\times180 + 2.5e^{-360} - 0 - \frac{5}{2}e^0$

$\doteqdot 897.5$

$\doteqdot 898$ metres deep.

Q7 (a) (i) In Δ^s XCY and XBY,

$\angle X$ is common

$\angle CYX = \angle CBY$ ($\angle$ in the alternate segment thm)

$\therefore \Delta XCY \,|||\, \Delta XBY$ (equiangular)

(ii) $\frac{XY}{BX} = \frac{CX}{XY}$ because they are pairs of corresponding sides in the similar Δ^s in part (i).

(iii) $XY = AX$ (given)

$\therefore \frac{AX}{BX} = \frac{CX}{AX}$

also, in Δ^s AXC and AXB,

$\angle X$ is common,

$\therefore \Delta AXC \,|||\, \Delta AXB$ (2 pairs of sides in same ratio and the included angle is equal)

(iv) $\angle ACX = \angle BAX$

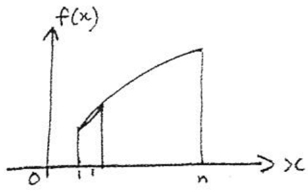
(Corresp. angles in similar Δ^s)

$\angle BDE = \angle ACX$ (ext $\angle$ of a cyclic quad. = opp. interior $\angle$)

$\therefore \angle BAX = \angle BDE$

$\therefore$ DE $\parallel$ AX (corresp. $\angle^s$ are equal)

(b) (i)



(ii) $\int_1^n f(x)\,dx > \frac{1}{2}\times1\times(f(1)+f(2)) + \frac{1}{2}\times1\times(f(2)+f(3))$

$+ \cdots + \frac{1}{2}\times1\times(f(n-1)+f(n))$

$= \frac{1}{2}[f(1)+f(n)] + f(2)+f(3)+\cdots+f(n-1)$

$= \frac{1}{2}f(1) + \frac{1}{2}f(n) + \sum_{r=1}^{n-1} f(r)$

7b) (iii) Let $f(x) = \log_e x$ in (ii)

$$\int_1^n \log_e x \, dx > \tfrac{1}{2}\ln 1 + \tfrac{1}{2}\ln n + \ln 2 + \ln 3 + \dots + \ln(n-1)$$
$$= 0 + \tfrac{1}{2}\ln n + \ln 2\times 3\times \dots \times (n-1)$$

Now Let $I = \int_1^n \log_e x \, dx$ $\quad u = \log_e x \quad dv = dx$ $\quad du = \frac{1}{x}dx \quad v = x$

$$\therefore \int_1^n \log_e x \, dx = \left[x \log_e x\right]_1^n - \int_1^n x \times \tfrac{1}{x}\, dx$$
$$= n\log_e n - 1\times\log_e 1 - \int_1^n dx$$
$$= n\log_e n - [x]_1^n$$
$$= n\log_e n - n + 1$$

$$\therefore n\log_e n - n + 1 > \tfrac{1}{2}\log_e n + \log_e (n-1)!$$
$$\ln n^n - n + 1 > \ln n^{1/2} + \ln(n-1)!$$
$$\therefore -n+1 > \ln n^{1/2} + \ln(n-1)! - \ln n^n$$
$$= \ln\left(\frac{n^{1/2}\times(n-1)!}{n^n}\right)$$

$$\therefore e^{-n+1} > \frac{n^{1/2-n}}{n} \times n!$$
$$= n^{-1/2-n} \times n!$$
$$n! < e^{1-n} \times n^{1/2+n}$$
$$\therefore n! < n^{n+1/2} \cdot e^{-n+1}$$

Q8 (a) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

at $P(a\cos\theta, b\sin\theta)$, m. of tang:

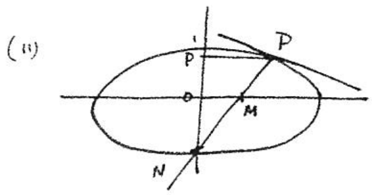
$$\frac{2a\cos\theta}{a^2} + \frac{2b\sin\theta}{b^2} \times \frac{dy}{dx} = 0$$
$$\frac{2\cos\theta}{a} + \frac{2\sin\theta}{b} \times \frac{dy}{dx} = 0$$
$$\therefore \frac{dy}{dx} = -\frac{\cos\theta}{a} \times \frac{b}{\sin\theta}$$

$\therefore$ m of normal at P $= \frac{a\sin\theta}{b\cos\theta}$

$\therefore$ eq'n of normal at P is:

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

(ii)



$\frac{PM}{PN} = \frac{P'O}{P'N}$ (ratio of intercepts on parallel lines)

$P'(0, b\sin\theta)$

at N, $x = 0$ on normal PN

$$\therefore y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}\, x - a\cos\theta$$

$$y = b\sin\theta - \frac{a^2\sin\theta}{b}$$
$$= \frac{\sin\theta}{b}(b^2 - a^2)$$

$$\therefore N\left(0, \frac{\sin\theta}{b}(b^2-a^2)\right)$$

$$\therefore \frac{P'O}{P'N} = \frac{b\sin\theta}{b\sin\theta - \frac{\sin\theta}{b}(b^2-a^2)}$$
$$= \frac{b^2}{b^2-(b^2-a^2)}$$
$$= \frac{b^2}{a^2}$$
$$= 1 - e^2$$

$$\therefore \frac{PM}{PN} = 1 - e^2$$

8 (b) (i) Show $\frac{d}{dx}\left\{x^2A(x)\right\} = x(1+x)^n$

$$LHS = \frac{d}{dx}\left\{x^2\left(\frac{1}{2} + \frac{1}{3}\binom{n}{1}x + \frac{1}{4}\binom{n}{2}x^2 + \dots + \frac{1}{n+2}x^n\right\}$$

$$= \frac{d}{dx}\left\{\frac{x^2}{2} + \frac{1}{3}\binom{n}{1}x^3 + \frac{1}{4}\binom{n}{2}x^4 + \dots + \frac{1}{n+2}x^{n+2}\right\}$$

$$= \frac{2x}{2} + \frac{1}{3}\binom{n}{1}\times 3x^2 + \frac{1}{4}\binom{n}{2}\times 4x^3 + \dots + \frac{1}{n+2}\times(n+2)x^{n+1}$$

$$= x + \binom{n}{1}x^2 + \binom{n}{2}x^3 + \dots + x^{n+1}$$

$$RHS = x\left\{1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + x^n\right)$$

$$= x + \binom{n}{1}x^2 + \binom{n}{2}x^3 + \dots + x^{n+1}$$

$$= LHS$$

(ii) Show $x(1+x)^n = (1+x)^{n+1} - (1+x)^n$

$$RHS = (1+x)^n(1+x-1)$$

$$= x(1+x)^n$$

$$= LHS$$

(iii) $\therefore$ From (i) $\frac{d}{dx}\left\{x^2A(x)\right\} = (1+x)^{n+1} - (1+x)^n$

$\int$ b.s. $\quad x^2A(x) = \frac{(1+x)^{n+2}}{n+2} - \frac{(1+x)^{n+1}}{n+1} + C$

Let $x=0$ $\quad 0 = \frac{1}{n+2} - \frac{1}{n+1} + C$

$$\therefore C = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\therefore x^2A(x) = \frac{(1+x)^{n+2}}{n+2} - \frac{(1+x)^{n+1}}{n+1} + \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{(1+x)^{n+2} - 1}{(n+2)} - \frac{(1+x)^{n+1} - 1}{(n+1)}$$

(iv) Let $x=1$ in (iv)

$$A(1) = \frac{2^{n+2} - 1}{n+2} - \frac{2^{n+1} - 1}{n+1}$$

Also let $x=1$ in original eq'n

$$A(1) = \frac{1}{2} + \frac{1}{3}\binom{n}{1} + \frac{1}{4}\binom{n}{2} + \dots + \frac{1}{n+2}\binom{n}{n}$$

$$\therefore \frac{1}{2} + \frac{1}{3}\binom{n}{1} + \frac{1}{4}\binom{n}{2} + \dots + \frac{1}{n+2}\binom{n}{n} = \frac{2^{n+2} - 1}{n+2} - \frac{2^{n+1} - 1}{n+1}$$

$$\therefore \sum_{r=0}^{n} \frac{1}{r+2}\binom{n}{r} = \frac{(2^{n+2}-1)(n+1) - (2^{n+1}-1)(n+2)}{(n+2)(n+1)}$$

$$= \frac{2^{n+2}(n+1) - (n+1) - 2^{n+1}(n+2) + (n+2)}{(n+2)(n+1)}$$

$$= \frac{2^{n+1}\times(2n+2) - 2^{n+1}(n+2) - n - 1 + n + 2}{(n+2)(n+1)}$$

$$= \frac{2^{n+1}(n) + 1}{(n+1)(n+2)}$$